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NON-STEADY WAVES IN AN ACOUSTIC MEDIUM WITH BOUNDARIES SHAPED LIKE A PARABOLIC CYLINDER*

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A method is proposed for solving certain two-dimensional non-steady-state problems for an acoustic medium outside a parabolic cylinder. In all cases the problem reduces to solving a Volterra integral equation of the second kind.

The short-wave asymptotic behaviour of a diffraction field in an acoustic medium was studied in /1, 2/. An analytical solution for the diffraction of non-steady waves was derived in /3, 4/ only for two-dimensional waves with a front perpendicular to the plane of symmetry of the cylinder.

1. Statement of the problem and the method of solving it. We introduce non-dimensional variables

$$X=\frac{X'}{A},\quad Y=\frac{Y'}{A},\quad t=\frac{t'c}{A},\quad \varphi=\frac{\varphi'}{Ac},\quad p=\frac{p'}{\rho c^2},\quad p=-\frac{\partial\varphi}{\partial t}$$

Throughout, all primed variables are dimensional; X', Y' are Cartesian coordinates, A is a characteristic linear dimension, c is the velocity of propagation of longitudinal waves, ρ is the density of the medium, p' is the pressure, and φ' is the velocity potential of reflected waves.

The medium will be considered in a parabolic system of coordinates ξ, η, so that the curve $\eta=\eta_0$ defines a parabola, and then

$$X=(\xi^2-\eta^2)/2,\ Y=\xi\eta,\ \varepsilon=\eta_0^2/2$$

where ε is the focal length. As a result we obtain the following problem for the potential:

$$\begin{gathered}\partial^2\varphi/\partial\xi^2+\partial^2\varphi/\partial\eta^2=(\xi^2+\eta^2)\,\partial^2\varphi/\partial t^2\\ \varphi\,|_{t=0}=\partial\varphi/\partial t\,|_{t=0}=0,\ \lim_{\eta\to\infty}\varphi=0\\ (\varkappa\varphi+\partial\varphi/\partial\eta)_{\eta=\eta_0}=Q(\xi,\ t)\end{gathered}\tag{1.1}$$

where $\varkappa$ is a parameter; $\varkappa=0$ for diffraction by an absolutely rigid surface, and $\varkappa=\infty$ for diffraction by an absolutely soft surface. The function Q characterizes the inhomogeneity of the boundary condition; in diffraction problems it has the following form (φ_0 denotes the potential of the incident wave):

$$Q(\xi,\ t)=-(\varkappa\varphi_0+\partial\varphi_0/\partial\eta)_{\eta=\eta_0}\tag{1.2}$$

Evaluating the Laplace transform of the solution with respect to the time variable and separating the space variables in the wave Eq.(1.1), we obtain

$$\begin{gathered}\varphi^L(\xi,\eta,s)=\sum_{n=0}^{\infty}\Phi_n{}^L(\eta,s)\,B_n{}^L(\xi,s)\\ \partial^2B_n{}^L/\partial\xi^2+(c_n-s^2\xi^2)\,B_n{}^L=0\\ \partial^2\Phi_n{}^L/\partial\eta^2-(c_n+s^2\eta^2)\,\Phi_n{}^L=0\end{gathered}\tag{1.3}$$

where the index L indicates transforms, s is the transform parameter and c_n the separation constants.

Eqs.(1.3) have single-valued solutions /5/ for the following values of c_n:

$$c_n=(2n+1)\,s,\ n=0,\ \pm1,\ \pm2,\ldots\tag{1.4}$$

It has been shown, relying on the asymptotic behaviour of the Weber functions /6/, that

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the solution of (1.3) for diverging waves must have the form

$$\varphi^L(\xi, \eta, s) = \sum_{n=0}^{\infty} E_n(s) D_n(\xi \sqrt{2s}) D_{-n-1}(\eta \sqrt{2s}) \tag{1.5}$$

Here $D_\nu(z)$ are Weber functions (of a parabolic cylinder). We recall that Weber functions of non-negative integer orders form a complete orthogonal system in $L_2(-\infty, +\infty)$:

$$\int_{-\infty}^{+\infty} D_n(z) D_m(z)\, dz = n! \sqrt{2\pi}\, \delta_{n,m} \tag{1.6}$$

and have the following representation /7/:

$$D_n(z\sqrt{2}) = \exp\left(-\frac{z^2}{2}\right) z^{\gamma_n} \sum_{k=0}^{\lambda_n} A_{n,k} z^{2k} \tag{1.7}$$

$$A_{nk} = (n+\gamma_n-1)!!\,(-1)^{\lambda_n-k}\lambda_n!\; 2^{k+\gamma_n/2}\,[(2(k+\gamma_n)-1)!!\,(\lambda_n-k)!\,k!]^{-1}$$
$$\lambda_n = [n/2],\quad \gamma_n = n - 2\lambda_n$$

In view of the difficulties involved in inverting expressions whose denominators contain the functions $D_{-n-1}(\eta_0\sqrt{2s})$ and their derivatives, we shall not look for the transforms of the solutions in explicit form. Using (1.4) and the zero initial data, we will use (1.3) to obtain hyperbolic equations for the inverse transforms of B_n^L, Φ_n^L:

$$\begin{aligned} &\partial^2 B_n/\partial\xi^2 + (2n+1)\,\partial B_n/\partial t - \xi^2\partial^2 B_n/\partial t^2 = 0 \\ &\partial^2\Phi_n/\partial\eta^2 - (2n+1)\,\partial\Phi_n/\partial t - \eta^2\partial^2\Phi_n/\partial t^2 = 0 \end{aligned} \tag{1.8}$$

We will first consider the solutions of the first equation in (1.8). Transforming to characteristic variables $x = t - \xi^2/2$, $y = t + \xi^2/2$, we obtain equations of the Euler-Darboux type. Using the substitution

$$B_n = (y-x)^{(1-\gamma_n)/2}\partial^{\lambda_n} Z_n(x, y)/\partial x^{\lambda_n}$$

we can show that the solution bounded at $\xi = 0$ is

$$B_n(\xi, t) = \sum_{k=0}^{\lambda_n} A_{n,k}\xi^{2k+\gamma_n} \frac{d^k b_n(t-\xi^2/2)}{dt^k} \tag{1.9}$$

Here b_n is an arbitrary function that vanishes for positive values of the argument. From (1.7) and (1.9) we can conclude that the Laplace transform of $B_n(\xi, t)$ is

$$B_n^L(\xi, s) = b_n^L(s) D_n(\xi\sqrt{2s})\, s^{-\gamma_n/2} \tag{1.10}$$

This result is in complete agreement with the formula for the transform of the general solution (1.5).

To solve the second equation in (1.8) we assume that the boundary values of the unknown function are known:

$$\Phi_n(\eta_0, t) = d_n(t),\quad \partial\Phi_n(\eta_0, t)/\partial\eta = q_n(t) \tag{1.11}$$

Transforming in the equation for Φ_n (1.8) to characteristic variables $x = t - \eta^2/2$, $y = t + \eta^2/2$, we again obtain equations of the Euler-Darboux type, for which we can write down the Riemann function /8/:

$$V_n(x, y, x_0, y_0) = \frac{(y-x)^{1/2}(y-x_0)^{n/2}}{(y_0-x)^{(n+1)/2}} F\left(\frac{n+1}{2}, -\frac{n}{2}, 1, \sigma\right) \tag{1.12}$$
$$\sigma = (x-x_0)(y-y_0)/[(y-x_0)(x-y_0)]$$

where $F(\alpha, \beta, \gamma, \sigma)$ is the hypergeometric function. Using the latter's properties /7/, we can represent the Riemann function (1.12) as follows ($P_m^{(\alpha,\beta)}$ are the Jacobi polynomials):

$$V_{2m}(x, y, x_0, y_0) = \frac{(y-x)^{1/2}(y-x_0)^m}{(y_0-x)^{m+1/2}} P_m^{(0,\,-1/2)}(1-2\sigma) \tag{1.13}$$

$$V_{2m+1}(x, y, x_0, y_0) = \frac{(y-x)(y_0-x_0)^{1/2}(y-x_0)^m}{(y_0-x)^{m+3/2}} P_m^{(0,\,1/2)}(1-2\sigma)$$

In accordance with Riemann's method /8/, the solution of the second equation of (1.8) in the region $t \geqslant 0, \eta \geqslant \eta_0$ satisfying the zero initial conditions may be written as follows (Fig.1):

$$\Phi_n(z, t) = \frac{1}{2}\left[\left(\frac{z_0}{z}\right)^{(n+1)/2} d_n(t-z+z_0) - \int_0^{t-z+z_0} (d_n(\tau) A_n(t, \tau, z, z_0) + \right. \tag{1.14}$$

$$\left. q_n(\tau) B_n(t, \tau, z, z_0))\, d\tau \right], \quad B_n(t, \tau, z, z_0) =$$

$$(2z_0)^{-1/2} V_n(\tau - z_0, \tau + z_0, t - z, t + z), \quad A_n(t, \tau, z, z_0) =$$

$$-(2z_0)^{1/2} \partial B_n / \partial z_0, \quad z = \eta^2/2, \quad z_0 = \eta_0^2/2$$

Putting $z = z_0$ in (1.14), we obtain an integral equation for the boundary values:

$$d_n(t) + \int_0^t [\alpha_n(t-\tau) d_n(\tau) + \beta_n(t-\tau) q_n(\tau)]\, d\tau = 0 \tag{1.15}$$

$$\alpha_n(t) = A_n(t, 0, z_0, z_0), \ \beta_n(t) = B_n(t, 0, z_0, z_0)$$

Later we shall need explicit expressions for the integral kernels in (1.15) for even n, which may be obtained from (1.13)-(1.15):

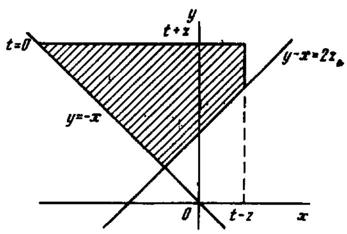

Fig.1

$$\beta_{2m}(t) = \frac{(\eta_0^2 - t)^m}{(\eta_0^2 + t)^{m+1/2}} P_m^{(0,\,-1/2)}(\lambda) \tag{1.16}$$

$$\alpha_{2m}(t) = \frac{(4m+1)t - \eta_0^2}{2(t^2 - \eta_0^4)} \eta_0 \beta_{2m}(t) -$$

$$\frac{\eta_0^3 (\eta_0^2 - t)^{m-1}}{(\eta_0^2 + t)^{m+3/2}} \times$$

$$m t^2 (2m+1) P_{m-1}^{(1,\,1/2)}(\lambda), \quad \lambda = \frac{t^2}{t^2 - \eta_0}$$

It now remains to pick the functions d_n, g_n in such a way that φ satisfies the boundary condition (1.1). Using (1.6), we expand the transform of $Q(\xi, t)$ in a series of Weber functions:

$$Q^L(\xi, s) = \sum_{n=0}^{\infty} Q_n^L(s) D_n(\xi \sqrt{2s}), \tag{1.17}$$

$$Q_n^L(s) = \frac{1}{n! \sqrt{\pi}} \int_{-\infty}^{+\infty} \Omega^L(w, s) D_n(w\sqrt{2})\, dw, \quad \Omega^L(w, s) = Q^L(\xi, s)|_{\xi = w/\sqrt{s}}$$

By the convolution theorem, Eq.(1.15) may be written as follows in the transform space:

$$d_n^L(s)(1 + \alpha_n^L(s)) + q_n^L(s) \beta_n^L(s) = 0 \tag{1.18}$$

The transforms of the boundary values of the unknown function are, by (1.3), (1.10), (1.11),

$$\varphi^L(\xi, \eta_0, s) = \sum_{n=0}^{\infty} U_n^L(s) D_n(\xi\sqrt{2s}) \tag{1.19}$$

$$\frac{\partial \varphi^L(\xi, \eta_0, s)}{\partial \eta} = \sum_{n=0}^{\infty} G_n^L(s) D_n(\xi \sqrt{2s})$$

$$U_n^L = b_n^L d_n^L s^{-\gamma_n/2}, \quad G_n^L = b_n^L q_n^L s^{-\gamma_n/2}$$

Formulae (1.1), (1.17)-(1.19) now yield the following system of equations for the transforms of U_n, G_n:

$$U_n^L(s)(1 + \alpha_n^L(s)) + G_n^L(s) \beta_n^L(s) = 0, \quad \varkappa U_n^L(s) + G_n^L(s) = 0 \tag{1.20}$$

If $\varkappa \neq \infty$ in (1.1) then, eliminating G_n^L, we obtain a Volterra equation of the second kind for U_n^L:

$$U_n(t) + \int_0^t U_n(t-\tau)(\alpha_n(\tau) - \varkappa\beta_n(\tau))\,d\tau = -\int_0^t \beta_n(\tau) Q_n(t-\tau)\,d\tau \tag{1.21}$$

$$Q_n(t) = \frac{1}{n!\sqrt{\pi}} \int_{-\infty}^{+\infty} \Omega(w, t) D_n(w\sqrt{2})\,dw$$

An expression for $Q_n(t)$ is obtained from (1.17) by inverting the order of the inverse Laplace transformation and integrating with respect to w, which is possible provided the integral with respect to s is uniformly convergent in some right half-plane. The inverse transform of $\Omega^L(w, s)$ may be written explicitly in various specific cases.

If $\varkappa = \infty$ the functions $U_n(t)$ are known and (1.20) yields an equation for $G_n(t)$:

$$\frac{1}{\eta_0} G_n(t) + \int_0^t \beta_n'(t-\tau) G_n(\tau)\,d\tau = \tag{1.22}$$

$$- U_n'(t) - \int_0^t \alpha_n'(t-\tau) U_n(\tau)\,d\tau - \alpha_n(0) U_n(t)$$

Once U_n and G_n have been determined from (1.21) or (1.22), the transform of the potential at an arbitrary point may be determined from (1.3), (1.10), (1.14) and (1.19):

$$\varphi^L(\xi, \eta, s) = \frac{1}{2}\exp\Big(-(\eta^2 - \eta_0^2)\frac{s}{2}\Big) \sum_{n=0}^{\infty} \Big\{U_n^L(s)\Big[\Big(\frac{\eta_0}{\eta}\Big)^{2(n+1)} - \tag{1.23}$$

$$A_n^L(s, 0, \eta^2/2, \eta_0^2/2)\Big] - G_n^L(s) B_n^L(s, 0, \eta^2/2, \eta_0^2/2)\Big\} D_n(\xi\sqrt{2s})$$

However, the properties of the series (1.19), (1.23) are not such as to guarantee the legitimacy of term-by-term inversion, since in most cases they are not uniformly convergent with respect to s in the right half-plane. Indeed, each term in these series yields a delay factor $H(t - \xi^2/2)$ (where H is the Heaviside unit function), whereas the sum of the series has no such factor. Exceptions to this rule occur in the diffraction of a two-dimensional wave with front perpendicular to the axis of the parabola and also for the point $\xi = 0$ corresponding to the apex of the parabola. In the latter case it follows from (1.7), (1.19) that

$$\varphi(0, \eta_0, t) = \sum_{m=0}^{\infty} (2m-1)!!\,(-1)^m U_{2m}(t) \tag{1.24}$$

To find the boundary values of the potential at $\xi \neq 0$, we first consider the case of a problem exhibiting symmetry with respect to the axial plane of the parabolic cylinder. Then the only terms retained in the series of (1.19) are those with even indices. We will confine our attention to values of $\xi \geqslant 0$. Since the functions $D_{2k}(\sqrt{2w})$ form a complete orthogonal system in the space of functions integrable with weight $w^{-1/2}$ over the interval $(0, +\infty)$, it is possible, by expanding the functions $D_{2m}(w\sqrt{2})$ in terms of this system, to represent the right-hand side of (1.19) as a series in $D_{2k}(s^{1/4}\sqrt{2\xi})$:

$$\varphi^L(\xi, \eta_0, s) = \sum_{k=0}^{\infty} v_{2k}^L(s) D_{2k}(s^{1/4}\sqrt{2\xi}) \tag{1.25}$$

$$v_{2k}(t) = \sum_{m=0}^{\infty} c_{2m,2k} U_{2m}(t)$$

where $c_{2m,2k}$ are the coefficients of the expansion of $D_{2m}(w\sqrt{2})$ in terms of the system $D_{2k}(\sqrt{2w})$. Their values may be found from (1.7) and formulae in /9/:

$$c_{2m,2k} = \frac{\exp(1/16)}{(2k)!} \sum_{i=0}^{m} \sum_{j=0}^{k} A_{2m,i} A_{2k,j} \frac{(2(2i+j)-1)!!}{2^{2i+j}} D_{-2i-j-1/2}(1/2) \tag{1.26}$$

On the basis of the formulae of operational calculus /10/ and formulae (1.25), we obtain a representation of the potential on the boundary of the cylinder:

$$\varphi(\xi, \eta_0, t) = \sum_{k=0}^{\infty} \int_0^t v'_{2k}(\tau) R_{2k}(\xi, t-\tau)\, d\tau \tag{1.27}$$

$$R_{2k}(\xi, t) = L^{-1}[D_{2k}(s^{1/4}\sqrt{2\xi})\, s^{-1}] =$$

$$\sqrt{\frac{2}{\pi}} \sum_{j=0}^{k} \xi^j A_{2k,\, j} (2t)^{-j/2} \exp\left(-\frac{\xi^2}{32t}\right) D_{j-1}\left(\frac{\xi}{4\sqrt{t}}\right)$$

where L^{-1} is the inverse Laplace transform operator. When $\xi = 0$ formula (1.27) reduces to (1.24).

Proceeding in the same way for the series (1.23), we can find the value of the potential at an arbitrary point of the external medium.

If the potential is not an even function of ξ, one must consider the following complete system of orthogonal functions, which are integrable with weight $|w|^{-1/2}$ over the interval $(-\infty, +\infty)$:

$$r_{2k}(w) = D_{2k}(\sqrt{2\,|w|}) \tag{1.28}$$

$$r_{2k+1}(w) = \begin{cases} D_{2k}(\sqrt{2w}), & w \geqslant 0 \\ -D_{2k}(\sqrt{-2w}), & w < 0 \end{cases}$$

It is then possible to expand (1.19) anew in terms of the functions (1.28), proceeding in the same way as before.

2. *Examples.* We will now examine a series of specific problems concerning the diffraction of waves by the absolutely rigid surface of a parabolic cylinder and some other non-steady-state problems.

Diffraction of a step-shaped pressure wave with front perpendicular to the plane of symmetry of the cylinder. In this case the boundary condition (1.1) has the form

$$\partial\varphi(\xi, \eta_0, t)/\partial\eta = \eta_0 H(t - \xi^2/2) \tag{2.1}$$

$$Q^L(\xi, s) = \eta_0 D_0(\xi\sqrt{2s})/s,$$

The only term retained in expansion (1.17) is the term with $n = 0$. From (1.16) and (1.21) we obtain the equation

$$U_0(t) + \frac{\eta_0}{2}\int_0^t \frac{U_0(t-\tau)\, d\tau}{(\tau + \eta_0^2)^{3/2}} = -\eta_0 \int_0^t \frac{d\tau}{\sqrt{\tau + \eta_0^2}} \tag{2.2}$$

It now follows from (1.19) that the pressure at the surface of the cylinder depends only on the delay $t - \xi^2/2$, and using (2.2) one can show that U_0 depends on t/ε. From (2.2) one obtains an equation for the total pressure on the surface of the cylinder $\varepsilon = 1$:

$$p^{\Sigma}(t) + \int_0^t \frac{p^{\Sigma}(t-\tau)\, dt}{\sqrt{2}(\tau + 2)^{3/2}} = 2 \tag{2.3}$$

Eq.(2.3) is identical with that obtained in /2, 3/ by transforming to space-time variables.

Diffraction of a cylinder wave by the absolutely rigid surface of a parabolic cylinder. Let us assume that the source of the wave is at a point $M(-a, b)$, where $a > \varepsilon$, $b \geqslant 0$ (Fig. 2) and the potential of the incident wave has the Laplace transform

$$\varphi_0{}^L(\xi, \eta, s) = -W^L(s)\, K_0(sR) \tag{2.4}$$

$$R^2 = (\xi^2/2 + a + \eta^2/2)^2 - 2\xi\eta b + b^2 - 2a\eta^2$$

where K_ν is the modified Bessel function of second order, R is the distance from the source to the point under consideration and W^L characterizes the wave profile beyond the front. We shall assume that

$$W^L(s) = \sqrt{2(a-\varepsilon)/\pi}\, s^{-1/2} \tag{2.5}$$

corresponding to a pressure jump of $R^{-1/2}$ at the front, with the wave subsequently damped at any point of the space. Using the Duhamel integral one can also deal with any other variations of the pressure with time.

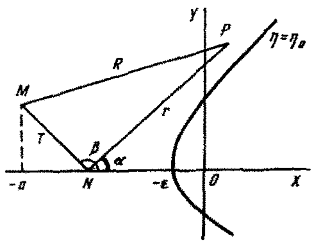

Fig.2

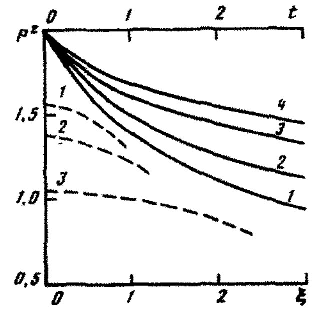

Fig.3

Substituting (2.3) into (1.2), (1.17), the problem reduces to inverting a Macdonald function with argument $(A_1s^2+bs^{3/4}+A_3s+A_4)^{1/2}$, $A_i=A_i(w)$, which is rather difficult to do by elementary methods if $b\neq 0$. Consider a point N on the axis of symmetry of the parabola, equidistant from the apex and the source, and an arbitrary point P.

Using the addition theorem for cylindrical functions /10/, we express the incident wave potential as a series

$$\varphi_0^L(\xi,\eta,s)=-ks^{-3/4}\sum_{n=0}^{\infty}e_nK_n(sr)I_n(sT)\cos(n\beta-n\alpha) \tag{2.6}$$

$$k=\left[\frac{2(a-\varepsilon)}{\pi}\right]^{1/2},\quad r^2=c^2+\left(\frac{\xi^2+\eta^2}{2}\right)^2+c(\xi^2-\eta^2)\equiv|PN|^2$$

$$e_n=\begin{cases}2, & n\neq 0\\ 1, & n=0,\end{cases}\quad c=\frac{b^2}{2(a-\varepsilon)}+\frac{a+\varepsilon}{2},\quad T=c-\varepsilon$$

where I_n is the modified Bessel function of first order and α and β the angles between the segments PN, MN and the X axis.

For the sequel we shall need the following trigonometric relations:

$$\cos\alpha=h/r,\quad \sin\alpha=\xi\eta/r,\quad h=(\xi^2-\eta^2)/2+c \tag{2.7}$$

$$\cos n\theta=\sum_{k=0}^{n}C_{nk}\cos^k\theta,\quad \sin n\theta=\sin\theta\sum_{k=0}^{n}B_{nk}\cos^k\theta$$

Here C_{nk} and B_{nk} are constants (see /10/).

Substitution of (2.6) into formula (2.5) and differentiation of the latter with respect to η leads to the following form of the boundary condition (1.1), (1.2):

$$\left.\frac{\partial\varphi}{\partial\eta}\right|_{\eta=\eta_0}=\frac{\eta_0}{\sqrt{s}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{1}E_{nk}I_n(sT)\frac{\xi^jh^k}{r^{k+j}}\left\{\left[K_{n-1}(sr)+\frac{n}{sr}K_n(sr)\right]\times\right. \tag{2.8}$$

$$\left.\frac{T-\xi^2/2}{r}+K_n(sr)\left[\frac{j}{2\varepsilon s}-\frac{k}{s(\xi^2/2+T)}-\frac{(k+j)(\xi^2/2-T)}{sr^2}\right]\right\}_{\eta=\eta_0},$$

$$E_{nkj}=\begin{cases}e_nC_{nk}\cos n\beta, & j=0\\ e_nB_{nk}\sin n\beta, & j=1\end{cases}$$

By (1.2), (1.21), and (2.7), the derivation of the integral equation requires finding the inverse transform of the function

$$\Omega^L(w,s)=\eta_0\sum_{n,k,j}E_{nkj}I_n(sT)e^{-sT}\frac{w^j\chi_+^k}{s^{(1-j)/2}\zeta^{k+j}}\times \tag{2.9}$$

$$\{[K_{n-1}(\zeta)e^{sT}+n\zeta^{-1}K_n(\zeta)e^{sT}]\chi_-\zeta^{-1}+K_n(\zeta)e^{sT}\times$$

$$\left[\frac{j}{2\varepsilon s}-\frac{k}{\chi_+}+\frac{(k+j)\chi_-}{\zeta^2}\right]\Big\},\quad \chi_\pm=sT\pm w^2/2$$

$$\zeta=T[(s+\gamma w^2)^2-\sigma^2w^4]^{1/2},\quad \sigma=\frac{\varepsilon\sqrt{c}}{2T^2},\quad \gamma=\frac{c+\varepsilon}{2T^2}$$

The factors in (2.8) must be regrouped in such a way that the inversion is readily carried out for each of them separately, using known formulae from /10/.

The radiation of a wave by a parabolic cylinder moving forward at a given velocity. Suppose that the velocity of the cylinder in a Cartesian system of coordinates has components

$V_x(t)H(t)$, $V_y(t)H(t)$. Then the boundary condition (1.1) is

$$\partial\varphi/\partial\eta|_{\eta=\eta_0} = (V_x(t)\cos\theta + V_y(t)\sin\theta)\sqrt{\xi^2+\eta_0^2}H(t) \quad (2.10)$$
$$\cos\theta = -2[2\varepsilon/(8\varepsilon+\xi^2)]^{1/2}, \quad \sin\theta = \xi(8\varepsilon+\xi^2)^{-1/2}$$

where θ is the angle between the normal vector to the cylinder $\eta=\eta_0$ and the X axis.

Under these conditions, we deduce from (1.1), (1.17), (2.9) and the formulae of the operational calculus /10/ that

$$\Omega(w,t) = F_1(w,t) * [{}^1/_2 wV_y(t) * (\pi t)^{-1/2} - \sqrt{2\varepsilon}V_x(t)] \quad (2.11)$$
$$F_1(w,t) = \delta(t) + 3w^2(16\varepsilon)^{-1}\exp(-5t_1)[I_1(3t_1) + I_0(3t_1)]$$
$$t_1 = w^2t(16\varepsilon)^{-1}$$

After solving the integral Eqs.(1.21) using (2.11), the pressure at the surface of the cylinder can be evaluated using (1.24) and (1.27).

Expansion of a parabolic cavity due to sudden application of pressure. Asume that the pressure is constant along the boundary of the parabolic cylinder, equal to $p(t)H(t)$. Then it follows from (1.1) and (1.17) (see /8/) that

$$\varkappa=\infty, \quad Q^L(\xi,s) = -\frac{p^L(s)\varkappa}{s}, \quad U_{2m}(t) = -\int_0^t \frac{p(\tau)\,d\tau}{2^m m!}$$

Eq.(1.22) gives $G_{2m}(t)$. The velocity of the leading point of the parabola $\xi=0$ is determined from (1.19):

$$V(0,\eta_0,t) \equiv \frac{1}{\eta_0}\frac{\partial\varphi(0,\eta_0,t)}{\partial\eta} = \frac{1}{\eta_0}\sum_{m=0}^{\infty} G_{2m}(t)(2m-1)!!(-1)^m$$

The diffraction of a cylindrical wave, from a source in the plane of symmetry of a parabolic cylinder, by an absolutely rigid surface. Suppose that the incident wave is described by Eqs.(2.4) and (2.5). The solution can be obtained by using formulae (2.6)-(2.9), after putting $b=0$. But the algebra will be far simpler if (2.4) is substituted directly into (1.2) and (1.17), and the result is

$$\Omega^L(w,s) = \sqrt{\frac{2T_1}{\pi}}\frac{K_1(u)}{u}\left(\sqrt{s}T_1 - \frac{w^2}{2\sqrt{s}}\right) \quad (2.12)$$
$$u = T_1[(s+\gamma_1w^2)^2 - \sigma_1w^4]^{1/2}, \quad T_1 = a-\varepsilon$$
$$\sigma_1 = \frac{\sqrt{a}\eta_0}{\sqrt{2}T_1^2}, \quad \gamma_1 = \frac{a+\varepsilon}{2T_1^2}$$

Since both functions are even with respect to ξ, Eq.(1.21) must be considered for only $n=2m$. One can then derive from (2.12) and the formulae in /10/ an explicit expression for the functions that determine the right-hand side of the integral Eq.(1.21).

$$Q_{2m}(t) = \frac{\sqrt{2}\eta_0}{\pi\sqrt{\pi T_1}}\int_0^t\frac{d\tau}{\sqrt{t-\tau}}\int_0^\infty \exp(-\gamma_1w^2T_1)G(\tau,w)D_{2m}(w\sqrt{2})\,dw \quad (2.13)$$
$$G(t,w) = \exp(-\gamma_1w^2t)[(t+T_1)(t^2+2tT_1)^{-1/2}\operatorname{ch}v - (1+2\gamma_1T_1)(2\sigma_1T_1)^{-1}\operatorname{sh}v,$$
$$v = \sigma_1w^2(t^2+2tT_1)^{1/2}$$

Based on the form of Eqs.(1.21) and (2.13), one can show that U_{2m} depends only on the quotients t/ε and a/ε. It is therefore sufficient to solve the problem for $\varepsilon=1$.

The Volterra Eq.(1.21) has been solved by means of Simpson's and the trapezoidal quadrature formulae. The time interval $[0,t_m]$ was divided into N_0 equal intervals and the values of the function were found at each point successively. It was assumed during the calculations that $t_m=3$ and $N_0=50$, and six terms were retained in each of the series (1.24), (1.27).

The solid curves in Fig.3 plot the total pressure at the apex of the parabola $\varepsilon=1$ for different positions of the source. The calculations were carried out using (1.24). The values of the pressure evaluated using five and six terms of the series differed by at most 0.1%, indicating rapid convergence of the series. When $a\geqslant 9$ the pressure values differ from the pressure in a two-dimensional wave by at most 3%. Curves *1-4* correspond to the following positions of the source: $a = 1.5, 2, 3$, and 9. As follows from physical considerations, at the initial instant of time the pressure at the leading point is doubled.

The dashed curves *1-3* in Fig.3 represent the pressure distribution over the surface of the parabolic cylinder at times $t = 0.6, 1.2$, and 3. The source of the cylindrical wave is in the plane of symmetry and $a = 2$. The calculations were carried out using four terms of the expansion (1.27).

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INHOMOGENEOUS ELASTIC STRUCTURES OPTIMAL IN STIFFNESS*

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The problem of maximizing the stiffness (of minimizing the work of the external forces) of an elastic structure in which the shear modulus is the control or, in the two-dimensional case, the plate thickness /1-3/ is considered. Point-by-point and integral constraints are imposed on the control. Necessary Weierstrass-Erdmann conditions and Weierstrass conditions are obtained that enable qualitative deductions to be made about the optimal solution. These deductions do not agree with the results in /4/ in which, it is true, a problem of mathematical physics is examined.

1. Formulation of the problem. Let R^N be an N-dimensional Euclidean space of vectors $\mathbf{x} = x_i\mathbf{e}_i$, where $\mathbf{e}_i$ are the unit vectors of a Cartesian system of coordinates (here and everywhere henceforth the Latin subscripts i, j, k, l, m, n run through values from 1 to N and summation from 1 to N is assumed over the repeated subscripts i, j, k, l, m, n in the products), Ω is the projection domain in R^N, and Γ is the boundary of Ω.

We will assume that the domain Ω can be filled by an elastic inhomogeneous material

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